The successive over-relaxation method

1. Use four steps of the successive over-relaxation method with $\omega = 1.06$ to approximate a solution to a system of linear equations $A\mathbf{u} = \mathbf{b}$ using the Jacobi and Gauss-Seidel methods (optional) where

$$A = \begin{pmatrix} 5 & 1 & -2 \\ 1 & 10 & 2 \\ -2 & 2 & 10 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} -1.5 \\ 0.2 \\ -1.0 \end{pmatrix}.$$

Answer: Using the Jacobi method:

$$\mathbf{u}_0 = A_{\text{diag}}^{-1}\mathbf{b} = \begin{pmatrix} -0.3 \\ 0.02 \\ -0.1 \end{pmatrix}, \text{ and } \mathbf{u}_1 = \begin{pmatrix} -0.34664 \\ 0.073 \\ -0.16784 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -0.38384176 \\ 0.08914592 \\ -0.18489328 \end{pmatrix}, \mathbf{u}_3 = \begin{pmatrix} -0.39226318016 \\ 0.09573584672 \\ -0.19517979136 \end{pmatrix},$$

$$\mathbf{u}_4 = \begin{pmatrix} -0.39751644023168 \\ 0.09841386206208 \\ -0.19774500621696 \end{pmatrix}.$$

Using Gauss-Seidel (not required):

$$\mathbf{u}_0 = A_{\text{diag}}^{-1}\mathbf{b} = \begin{pmatrix} -0.3 \\ 0.02 \\ -0.1 \end{pmatrix}, \text{ and } \mathbf{u}_1 = \begin{pmatrix} -0.34664 \\ 0.07794384 \\ -0.19001177408 \end{pmatrix}, \mathbf{u}_2 = \begin{pmatrix} -0.3942906863 \\ 0.09860067845 \\ -0.1990922629 \end{pmatrix},$$

$$\mathbf{u}_3 = \begin{pmatrix} -0.3996610221 \\ 0.09985558737 \\ -0.1999519854 \end{pmatrix}, \mathbf{u}_4 = \begin{pmatrix} -0.3999693650 \\ 0.09999523836 \\ -0.1999953768 \end{pmatrix}.$$

2. The solution to Question 1 is the vector $\mathbf{u} = \begin{pmatrix} -0.4 \\ 0.1 \\ -0.2 \end{pmatrix}$. What is $\|\mathbf{u} - \mathbf{u}_k\|_2$ for each of these approximations?

Answer: For the Jacobi method:

0.1625, 0.06790, 0.02464, 0.01006, 0.003711

For the Gauss-Seidel method (not required):

0.1625, 0.05860, 0.005948, 0.0003716, 0.00003135

3. The errors in each approximation in Question 2 seem to drop by approximately a constant with each step. What would be your estimate as to the reduction in this error?

Answer: For the Jacobi method, the error drops by a factor of 3, while for Gauss-Seidel, the error appears to drop by a value around 10.

4. Verify your response to Question 3 by running the following Matlab code:

```
A = [5 1 -2; 1 10 2; -2 2 10];
b = [-1.5 0.2 -1.0]';
u = [-0.4 0.1 -0.2]';           # The exact solution to A*u = v
omega = 1.06;
Adiag = diag(diag(A));
Aoff  = A - Adiag;
u1 = inv(Adiag)*b;    % Jacobi
w1 = inv(Adiag)*b;    % Gauss-Seidel

for i = 1:5
  u0 = u1;
  w0 = w1;

  u1 = (1 - omega)*u1 + omega*inv(Adiag)*(b - Aoff*u1);

  for j = 1:3
    w1(j) = (1 - omega)*w1(j) + omega*(b(j) - Aoff(j,:)*w1)/Adiag(j,j);
  end

  [norm( u1 - u ) norm( u0 - u )/norm( u1 - u )]
  [norm( w1 - u ) norm( w0 - u )/norm( w1 - u )]
end
```

5. Demonstrate that using $\omega = 2$ allows Newton's method to converge with a rate of $O(h^2)$ as opposed to just $O(h)$ by showing a sufficient number of steps using Newton's method in finding the one root of the function $f(x) = e^{2x} - 6e^x + 9$. Recall that Newton's method is

$$x_{k+1} \leftarrow x_k - \frac{f(x_k)}{f^{(1)}(x_k)}$$

so using $\omega = 2$ results in

$$x_{k+1} \leftarrow 2\left(x_k - \frac{f(x_k)}{f^{(1)}(x_k)}\right) + (1-2)x_k = x_k - 2\frac{f(x_k)}{f^{(1)}(x_k)}$$

Answer: Using Newton's method (first column) and its absolute error (second column), we note that the error seems to drop by approximately a factor of two with each iteration, but using over-relaxation results in quadratic convergence (third and fourth columns):

1	0.09861	1	0.09861
1.051819161757164	0.04679	1.103638323514329	0.005026
1.075771763391340	0.02284	1.098624898047284	0.00001261
1.087323447096480	0.01129	1.098612288730268	0.00000000006216
1.092999847592984	0.005612	1.098612288730268	0.00000000006216
1.095813957757434	0.002798		
1.097215082704145	0.001397		
1.097914173959751	0.0006981		
1.098263353183153	0.0003489		
1.098437851368171	0.0001744		
1.098525077624158	0.00008721		
1.098568685044938	0.00004360		
1.098590487337610	0.00002180		
1.098601388111884	0.00001090		
1.098606838432560	0.000005450		

1.098612288668110 = ln(3)

The last iteration of Newton's method with over-relaxation is unchanged.